JOURNAL OF COMBINATORIAL THEORY, Series A 34, 60–70 (1983)

Decomposition of a Complete Multigraph into Simple Paths: Nonbalanced Handcuffed Designs

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Communicated by the Managing Editors

Received October 29, 1981

The condition $\lambda v(v-1) \equiv 0 \pmod{2m}$, $v \geqslant m+1$ is obviously necessary for the existence of an edge disjoint decomposition of a complete multigraph λK_v into isomorphic simple paths consisting of m edges each. This condition is proved here to be sufficient. The proof is also valid in some cases when the given paths are not necessarily of the same length.

INTRODUCTION

A complete multigraph λK_v is a complete graph K_v in which every edge is taken λ times. The multigraph λK_v is said to have a G decomposition $G[\lambda, v]$ if it is an edge-disjoint union of subgraphs all isomorphic to a fixed graph G. The basic problem related to this definition is to determine, for a given graph G, the necessary and sufficient conditions on λ and v for the existence of $G[\lambda, v]$. A partial list of known results and a description of methods used in this area may be found in [2].

In this paper we solve the problem for the case $G = P_m$, a simple path with m edges. The question of decomposing a complete graph into paths was first mentioned in [3] and then was formally stated as the *handcuffed prisoners problem* in [4]. The handcuffed prisoners problem is slightly different from the $P_m[\lambda, v]$ problem. In a handcuffed design, it is also required that every vertex should be contained in the same number of paths. Such a design is called a *balanced* $P_m[\lambda, v]$. The necessary condition for the existence of a handcuffed design is thus stronger than the condition for the existence of a $P_m[\lambda, v]$. Partial results for the balanced case were given in [4, 6, 8, 9]. It was finally solved by Huang [5] and by Hung and Mendelsohn [7]. Huang also solved the nonbalanced problem for $v \geqslant m^2$, (the problem becomes harder when $v - m$ is small). (From the referee of this

0097-3165/83/010060-11$03.00/0

paper I learned that M. Kovider was independently working on the nonbalanced case and solved it partially.) In this paper, we show that a $P_m[\lambda, v]$ exists if and only if $\lambda v(v-1) \equiv 0 \pmod{2m}$ and $v \geqslant m+1$. We also use the method of the proof to decompose a complete multigraph into a given sequence of simpe paths which are not necessarily of the same length.

2. Notations and Definitions

In most of the following constructions the set of vertices of K_v will be $V = Z_v$, the cyclic group with v elements (some times $Z_{v-1} \cup \{\infty\}$). Addition of vertices and multiplication of a vertex by a natural number will be frequently mentioned. These operations are mod v (or mod $v-1$). The notation D_v, the set of differences mod v, will also be used: If x and y are two elements of z_v, then the edge (x, y) is of difference $k \in D_v$ ($1 \leqslant k \leqslant v/2$ for v even, or $1 \leqslant k \leqslant (v-1)/2$ for v odd), where k is the *absolute difference* between x and y (more about difference sets can e found in [2]).

The general graph theoretical terminology used in this paper is taken from Berge [1]. For our specific subject we add the following notations:

(N1) the length $l(A)$ of a path A is the number of *edges* it contains.

(N2) the diameter of a nonsimple path $d(A)$ is the least number of edges between two appearances of the same vertex along A. (The length of the minimal cycle it contains.)

(N3) The path $\{(x_0, x_1), (x_1, x_2) \cdots (x_{n-1}, x_n)\}$ is briefly denoted by $\langle x_0, x_1, x_2, \ldots, x_n \rangle$.

(N4) For x_i, $k \in Z_v$, $A = \langle x_0, x_1, \ldots, x_n \rangle$, we define $A + k = \langle x_0 + k, x_1 + k, \ldots, x_n + k \rangle$ (addition mod v).

(N5) For a set $\mathcal{A}$ of paths $\mathcal{A} + k = \{A + k \mid A \in \mathcal{A}\}$.

(N6) For $A = \langle x_0, x_1, \ldots, x_{n-1}, x_n \rangle$

$$A^t = \langle x_n, x_{n-1}, \ldots, x_1, x_0 \rangle.$$

(N7) If $A = \langle x_0, x_1, \ldots, x_n \rangle$, $B = \langle y_0, y_1, \ldots, y_m \rangle$, and $x_n = y_0$, then

$$A + B = \langle x_0, x_1, \ldots, x_n, y_1, y_2, \ldots, y_m \rangle.$$

A and B are called segments of $A + B$. More than two segments can be concatenated the same way.

(N8) If A is a path for which the two end vertices are the same, then

$$nA = \overbrace{A + A + A \cdots + A}^{n \text{ times}}.$$

(N9) Let T be a path and m a natural number. By T/m we denote the set of paths obtained by cutting T, starting from its beginning, into paths of length m. If $l(T)$ is not an integer multiplication of m, then T/m contains a final segment of T, the length of which is less than m. This segment is called the remainder of T/m. Note: If $d(T) > m$, all the paths contined in T/m are simple.

(N10) $\mathbf{P_m}[\lambda, v]$: A decompoition of λK_v into simple paths p_m with a remainder is a set of simple paths all of them of length m, except one which is shorter. Every edge of K_v appears in exactly λ of those paths. The shorter path is called the remainder. Note that whenever $\lambda v(v-1) \equiv 0 \pmod{2m}$, every $\mathbf{Pm}[\lambda, v]$ is a $P_m[\lambda, v]$.

Now we state the main theorem.

THEOREM 1. *A necessary and sufficient condition for the existence of a* $P_m[\lambda, v]$, *a decomposition of a complete multigraph* λKv *into edge disjoint simple paths of length* m *is*

$$\lambda v(v-1) \equiv 0 (\bmod 2m) \qquad \text{and} \qquad v \geqslant m+1. \tag{1}$$

The necessity of (1) is obvious. In order to prove the sufficiency we use the following method: If λ is even or v is odd, then λKv is an Eulerian graph for which we construct an Eulerian path P with $d(P) = v-2$. Thus, for $m \leqslant v-3$, P/m is a $\mathbf{P_m}[\lambda, v]$, which is a $P_m[\lambda, v]$ when (1) holds. The cases $m = v-2$, $m = v-1$ are treated separately in a similar way. For even v and odd λ we construct at first, a set of $v/2$ paths, such that every vertex is an endpoint of exactly one of them. After taking these paths off, the remaining is an Eulerian graph. For this graph we construct an Eulerian path with diameter greater than m and divide it by m to obtain a $\mathbf{Pm}[\lambda, v]$. This process varies slightly while various values of v, m, and λ are considered. Actually, we have 16 variations of this main idea for 16 different cases.

In these variations we use the same fundamental constructions (as follows) based on the idea of differences sets.

$I(x, y)$

For $x, y \in Z_v$, $I(x, y) = \langle x, x+1, x+2, \ldots, y-1, y \rangle$.

$S(v, x, a, b)$

For $x \in Z_v$, $v/2 > b > a$, $b, a \in D_v$,

$$S(v, x, a, b) = \langle x, x+a, x-1, x+a+1, x-2, x+a+2, \ldots, y \rangle.$$

The vertex y is chosen to make the length of the path $b-a+1$ (the last edge is of difference b). $S(v, x, a, b)$ is a simple path containing exactly one edge of difference i for every $a \leqslant i \leqslant b$.

$L(v, x, a, b)$

For v even a, b and x as in the $S(v, x, a, b)$ construction

$$L(v, x, , a, b) = S(v, x, a, b) + \langle y, y + (v/2)\rangle + [S(v, x + (v/2), a, b)]'$$

y is the last vertex of $S(v, x, a, b)$. $L(v, x, a, b)$ is a simple path of length $2(b - a) + 3$. For every i, $a \leqslant i \leqslant b$, there are in the path two edges of difference i. It also contains one edge of difference $v/2$.

$M_{v,a,b}$

For v, a, and b satisfying the conditions of the previous construction

$$M_{v,a,b} = \{L(v, x, a, b) \mid 0 \leqslant x \leqslant (v/2) - 1\}.$$

This is a set of $v/2$ paths which together cover all the edges of differences a through b and $v/2$. Each vertex is the end point of exactly one path. The difference between the two ends of each path is $v/2$.

$C(v, k)$

For $k \in D_v$, $k \leqslant (v - 3)/2$

$$C(v, k) = \langle 0, k + 1, 1, k + 2, 2, k + 3, \ldots, k - 1, v - 1, k, 0\rangle.$$

$C(v, k)$ is a nonsimple path of length $2v$. It contains all the edges of differences k and $k + 1$.

$PC(v, a, b)$

For $a, b \in D_v$ such that $v/2 > b > a$ and $b - a$ is odd.

$$PC(v, a, b) = C(v, a) + C(v, a + 2) + C(v, a + 4) + \cdots + C(v, b - 1)$$

$l(PC(v, a, b)) = v(b - a + 1)$. The path contains all the edges of differences a–b.

An important quantity in the proof is the diameter of $PC(v, a, b)$: Looking at the structure of $C(v, k)$, one can verify that $d[C(v, k)] = 2k + 1$. Now, the way $C(v, k)$ and $C(v, k + 2)$ are concatenated makes the length of a cycle which begins in $C(v, k)$ and ends in $C(v, k + 2)$ equal to the length of a cycle which is entirely contained in either $C(v, k)$ or in $C(v, k + 2)$. Thus, we conclude $d[PC(v, a, b)] = 2a + 1$.

If $a \geqslant m/2$, then $d[PC(v, a, b)] > m$. As an immediate result we obtain the following:

LEMMA.

$$\text{If } a \geqslant m/2, \text{ then } PC(v, a, b)/m \text{ contains only simple paths.} \tag{2}$$

Before constructing $P_m[\lambda, v]$s for the various values of the parameters, let us mention the following:

LEMMA.

The existence of a $\mathbf{P_m}[\lambda', v]$ *is a sufficient condition for the existence of a* $P_m[\lambda, v]$ *for every* $\lambda = k\lambda'$ *for which* (1) *holds.* (3)

To prove this construct a $\mathbf{P_m}[\lambda', v]$. The set of vertices is Z_v; and label the vertices so that the remainder is $R_0 = I(0 \cdot r)$.

By $\mathcal{O}_0$ we denote the set of paths in this $\mathbf{P_m}[\lambda, v]$ without the remainder. We now define $\mathcal{O}_x = \mathcal{O}_0 + xr$ and $R_x = R_0 + xr$. The set $\bigcup_{x=0}^{k-1} \mathcal{O}_x \cup [(R_0 + R_1 + R_2 \cdots + R_{k-1})/m]$ is a $\mathbf{P_m}[\lambda, v]$ meaning a $P_m[\lambda, v]$ when (1) holds.

The proof will now be completed by listing the various constructions which cover all the possible values of v, m, and λ. The set of vertices is $V = Z_v$ unless it is defined otherwise. The validity of the constructions is based on the fact that T/m contains only simple paths for $m < d(T)$, and on Eqs. (2) and (3). In some cases, some technical effort is required to verify that $d(T)$ is really less than m and that all the edges are covered. This effort is referring to the definition of the fundamental constructions, paying attention to the notes given at the end of each definition, and evaluating some arithmetical inequalities. Since they are technical, but some times quite long, we deleted those details. One can see, however, the structure of those details following the proof for Case 16.

Case 1 (v odd, $m \leqslant v - 3$).

$$C_0 = \left\langle \infty, 0, 1, v-1, 2, v-2, ..., \frac{v-3}{2}, \frac{v+1}{2}, \frac{v-1}{2}, \infty \right\rangle,$$

$$C_x = C_0 + x(\infty + x = \infty).$$

The set $(C_0 + C_1, ..., + C_{(v-3)/2})/m$ is a $\mathbf{P_m}[1, v]$.

Case 2 (v ood, $m = v - 2$).

Define C_x as in Case 1 and $T_x = mC_x$. The set $\bigcup_{x=0}^{(v-3)/2} (T_x/m)$ is a $P_m[m, v]$ and (1) holds for these v and m only when $\lambda = km$.

Case 3 (v odd, $m = v - 1$).

$$P_0 = \langle 0, v-1, 1, v-2, 2, v-3, \ldots, (v+1)/2, (v-1)/2 \rangle,$$
$$P_x = P_0 + x.$$

The set $\{P_x \mid x \in Z_v\}$ is a $P_m[2, v]$ and (1) implies $\lambda = 2k$.

Case 4 (v even, $m = v - 1$).

$$M_{v,1,(m-1)/2} \quad \text{is a} \quad P_m[1, v].$$

Case 5 (v even, $v - m \equiv 1 (\text{mod } 4)$, $m \leqslant v - 5$).

$$M_{v,1,(m-1)/2} \cup PC(v, (m+1)/2, v/2 - 1)/m \quad \text{is a} \quad \mathbf{P_m}[1, v].$$

Case 6 (v even, $m = v - 3$).

$$M_{v,2,(m+1)/2} \cup \langle 0, 1, 2, \ldots, 0 \rangle/m \quad \text{is a} \quad \mathbf{P_m}[1, v].$$

Case 7 (v even, $v - m \equiv 3 (\text{mod } 4)$, $m \leqslant v - 7$).

$$M_{v,2,(m+1)/2} \cup \{I(0, m)\} \cup (I(m, 0) + PC(v, (m+3)/2, v/2 - 1))/m \quad \text{is a} \quad \mathbf{P_m}[1, v].$$

Case 8 (v even, $m = v - 2$).

Obtain M by adding an end edge out of $I(0, v/2)$ to each path from $M_{v,2,m/2}$. Now, $M \cup \{I(v/2, 0)\}$ is a $\mathbf{P_m}[1, v]$.

Case 9 (v even, $v - m \equiv 2$ (mod 4), $m \leqslant v/2$).

Define M as in Case 8.

$$M \cup (I(v/2, 0) + PC(v, m/2 + 1, v/2 - 1))/m \quad \text{is a} \quad \mathbf{P_m}[1, v].$$

Case 10 (v even, $v - m \equiv 2$ (mod 4), $v/2 < m \leqslant v - 6$). Define:

$$f = \min \left\{ \left[\frac{v}{2(v-m)} \right], \left[\frac{v(v-m-2)}{2(2m-v)} \right] \right\}$$

($[x]$ denotes the integer part of x),

$$S_i = I(m, 0) + (i-1)(m - v/2), \qquad 1 \leqslant i \leqslant f$$
$$S_R = I(0, v/2 - f(v-m)) + (f-1)v/2.$$

Denote by T_i, $1 \leqslant i \leqslant f$, f paths of length $2m - v$ each, cut along $PC(v, m/2 + 1,\ v/2 - 1)$. Denote by T_R, the final segment remaining of $PC(v, m/2 + 1, v/2 - 1)$ after taking out T_1–T_f.

Define now, for $1 \leqslant i \leqslant f$,

$$D_i = S_i + T_i \qquad \text{and} \qquad D_R = S_R + T_R.$$

Obtain M by adding an end edge of difference 1, taken from the edges which are not in $S_1, S_2, \dots, S_f, S_R$, to each path of $M_{v,2,m/2}$.

$$M \bigcup_{i=1}^{f} \{D_i\} \cup D_R/m \qquad \text{is a} \quad \mathbf{P}_m[1, v].$$

For the validity of this construction see Case 16.

Case 11 (v even, $v - m \equiv 0 \pmod 4$, λ even).

$$V = Z_{v-1} \cup \{\infty\}$$

$$C_0 = \langle \infty, 0, v-1, 1, v-2, 2, \dots, v/2, v/2-1, \infty \rangle$$

$$C_x = C_0 + x \ (\infty + x = \infty)$$

$$(C_0 + C_1 + C_2 + \cdots + C_{v-2})/m \qquad \text{is a} \quad \mathbf{P}_m[2, v].$$

Case 12 (v even, $v - m \equiv 0 \pmod 4$, λ odd, $m < v/2$).

A $\mathbf{P}_m[1, v]$ is obtained by cutting every path (except the remainder if $v < m$) of a $P_{2m}[1, v]$ into halves. (Eventually $2m$ will become greater than $v/2$.)

Case 13 ($m = 4$, $v = 8$).

$$v = Z_7 \cup \{\infty\}, \ \ \{\infty, x, x+6, x+1, x+5 \mid x \in Z_7\} \ \ \text{is a } P_4[1, 8].$$

Case 14 ($v - m \equiv 0 \pmod 4$, $v/2 \leqslant m < 3v/4$, $m \leqslant v - 8$, λ odd).

$$\begin{aligned} L'(v, x, 2, m/2) = {} & [S(v, x, 2, m/2)]' + \langle x, x + v/2 + 1, x + v/2 \rangle \\ & + S(v, x + v/2, 2, m/2), \\ M = {} & \{L'(v, x, 2, m/2) \mid 0 \leqslant x \leqslant v/2 - 1\}, \\ A = {} & I(0, v/2), \\ B = {} & \langle 0, v/2, 1, v/2 + 1, 2, v/2 + 2, 3, \dots, v-1, v/2 \rangle. \end{aligned}$$

Obtain $\bar{A}$ from A, replacing the last $(m - v/2)$ edges by the last $2(m - v/2)$ edges of B. Obtain $\bar{B}$ from B, replacing the last $2(m - v/2)$ edges by the $m - v/2$ last edges of A. If $m = v/2$, then $\bar{A} = A$, $\bar{B} = B$. Let C be the final segment of $\bar{B}$ containing m edges; D is what remains of $\bar{B}$ after taking C out.

$$M \cup \{\bar{A}, C\} \cup (PC(v, m/2 + 1, v/2 - 2) + D)/m \qquad \text{is a} \quad \mathbf{P}_m[1, v].$$

Case 15 (v even, $m = v - 4$, $m \geqslant 3v/4$, λ odd).

$$\begin{aligned} A = {} & \langle 3v/2 - m, v - m + 1, 3v/2 - m + 2, v - m + 3, \\ & \quad 3v/2 - m + 4, \ldots, v/2 - 1, 0\rangle + I(0, v - m) \\ & + \langle v - m, 3v/2 - m + 1, v - m + 2, \\ & \quad 3v/2 - m + 3, v - m + 4, \ldots, v - 1, v/2\rangle \\ B = {} & \langle v/2, 1, v/2 + 2, 3, v/2 + 4, 5, \ldots, v - m - 1, 3v/2 - m\rangle \\ & + I(3v/2 - m, 0) \\ & + \langle 0, v/2 + 1, 2, v/2 + 3, 4, \ldots, 3v/2 - m - 1, v - m\rangle. \end{aligned}$$

Obtain M from $M_{v,2,m/2}$ by adding an edge of difference 1, out of the edges which are not in $A \cup B$, to each path.

$$M \cup \{A, B\} \qquad \text{is a} \quad \mathbf{P_m}[1, v].$$

Case 16 (v even $v - m \equiv 0 \pmod 4$, $3v/4 \leqslant m \leqslant v - 8$).

The construction for this case is the most complicated as is the proof of its validity. Thus, we discuss this case in a wider form, including the technical details and explanations omitted in other sections. This case can be used as a guide for checking the validity of the other constructions.

Define A and B as in Case 15. Those definitions hold only if $v \equiv 0 \pmod 4$, which is satisfied by (1) for v even, $v - m \equiv 0 \pmod 4$, and λ odd.

From the definitions we obtain:

$$l(A) = m, \qquad l(B) = 3v/2 - m.$$

Both A and B are simple paths, together covering all the edges of difference $v/2 - 1$ and $v/2$ edges of difference 1. For the differences $m/2 + 1$ through $v/2 - 2$, we construct $T = PC(v, m/2 + 1, v/2 - 2)$. We still have the differences 2 through $m/2$ and $v/2$, covered in $M_{v,2,m/2}$. The paths of $M_{v,2,m/2}$ are of length $m - 1$ so we add to each of them an end edge of difference 1, out of $I(v - m, 3v/2 - m)$, which is still uncovered. This can be done since the end vertices of each path of $M_{v,2,m/2}$ are x and $x + v/2$. If $m = 3v/4$, we have $l(B) = m$, and we complete the construction by considering T/m, which according to (2) contains only simple paths.

If $m > 3v/4$, then B is too short. We start in this case by replacing the last $2m - 3v/2$ edges of the second segment of B, namely, the segment $I(5v/2 - 2m, 0)$, by the last $2(2m - 3v/2))$ edges of T which form the path

$$K = \langle x, y - 2, x + 1, y - 1, x + 2, y, x + 3, \ldots, v/2 - 3, 0\rangle,$$

where $x = 5v/2 - 2m$, $y = 2(v - m)$.

Let $\bar{B}$ denote the path obtained from B after this replacement $l(\bar{B}) =$

$3v/2 - m + (2m - 3v/2) = m$. The vertices not in $\bar{B}$ are those of $I(v-m+1, 2(v-m)-3)$ and $(v/2)-1$, $(v/2)-2$, all together $v-m-1$ vertices. Thus, the number of vertices in $\bar{B}$ is $m+1 = l(\bar{B})+1$ so $\bar{B}$ is simple.

We now define

$$f = \min\left\{\left[\frac{v}{2(v-m)}\right] - 2, \left[\frac{l(T\backslash K)}{2m-v}\right]\right\}.$$

This definition makes it possible to construct f paths T_i, $1 \leqslant i \leqslant f$, of length $2m-v$ each, along $T\backslash K$. By T_R we denote the final segment of $T\backslash K$ remaining after taking $T_1 \cdots T_f$ (maybe $T_R = \varnothing$). We also define S_i, $1 \leqslant i \leqslant f$, as $S_i = I(m, 0) + (i-1)(m-(v/2))$ and $S_R = I(2(v-m), v/2 - f(v-m)) + (f-1)v/2$. Together with the segments $I(0, v-m)$ contained in A and $I(3v/2 - m, 5v/2 - 2m)$ in $\bar{B}$, S_1 through S_f and S_R cover $v/2$ edges of difference 1, one of each pair $\langle x, x+1\rangle$, $\langle x + v/2, x + v/2 + 1\rangle$. Thus, the remaining edges of difference 1 can be added to the paths of $M_{v,2,m/2}$ to form the set M of $v/2$ simple path of length m each.

Notice that the last vertex of S_i, $(i-1)(m-(v/2))$, is the same as the first vertex of T_i and the last vertex of S_R is the first of T_R. Thus we define

$$D_i = S_i + T_i, \qquad 1 \leqslant i \leqslant f, \qquad D_R = S_R + T_R;$$

D_i are of length m. We now show that they are simple: The vertices in the odd places in T_i follow the consecutive vertices of S_i. The vertices in the even places are obtained by adding k, where $m/2 < k < v/2$.

If x is the common vertex of S_i and T_i, then: In the odd places of T_i, we have the vertices $x, x+1 \cdots x+m-v/2$, in the even places of T_i, the vertices are from the open interval $(x+(m/2), x+m)$ and in S_i, the vertices $x+m$, $x+m+1 \cdots x$. These three sets are disjoint (except x). Thus $S_i + T_i$ is simple.

Regarding D_R: If $f = [v/2(v-m)] - 2$, then $l(S_R) < l(S_i)$. Following the same argument as before, we obtain $d(D_R) > l(D_i) = m$.

If $f = [l(T\backslash K)/(2m-v)] - 2$, we substitute the lengths of T and K to get

$$(v(v/2 - m/2 - 2) - 4m + 3v)/(2m-v) < (v/2(v-m)) - 2.$$

Thus $(v-m-2)(v-m) < 2m-v$.

Regarding the identity:

$$\begin{aligned} l(T\backslash K) &= v((v-m-2/2) - 2(2m-v) \\ &= ((v-m-6/2)(2m-v) + (v-m-2)(v-m), \end{aligned}$$

we obtain $f = (v-m-6)/2$ and $r = l(T_R) = (v-m-2)(v-m) < 2m-v$ since $l(K) = 4m-3v$, $l(T_R+K) < 6m-4v < 2v$.

Thus, T_R is contained in the last circle of T, namely, $C(v, v/2-3)$; and T_R starts $l(K)+r$ edges before the end of T. Regarding the definition of T the first vertex of T_R is thus $x-r/2$, where x stands for $(5/2)v-2m$. In the odd places of T_R, the vertices are $x-r/2$ through x and in the even places $x-v/2-r/2-2$ through $x-v/2-2$. In this case, S_R is $I(2(v-m), x-r/2)$. The only common vertex of T_R and S_R is $x-r/2$, so D_R is a simple path. In either case, D_R/m contains only simple paths thus

$$\{D_1, D_2, \dots, D_f\} \cup \{A, \bar{B}\} \cup M \cup D_R/m \qquad \text{is a} \quad \mathbf{P}_{\mathbf{m}}[1, v].$$

A natural generalization of Theorem 1 is the following conjecture:

CONJECTURE. *Let $M=\{m_1, m_2, \dots, m_k\}$ be a sequence of natural numbers which satisfies $m_i \leqslant v-1$ for $1 \leqslant i \leqslant k$ and $\sum_{i=1}^{k} m_i = \lambda\binom{v}{2}$, then there exists a sequence of simple paths $P_{m_1}, P_{m_2}, \dots, P_{m_k}$ such that every edge of K_v belongs to exactly λ of them. Such a sequence is called a $P_M[\lambda, v]$.*

This conjecture is supported by checking all the possible sequences for small values of v (with $\lambda=1$) and by the following partial result:

THEOREM 2. *Let v be odd or λ even, and $M=\{m_1, m_2, \dots, m_k\}$ a sequence of natural numbers with*

$$m_i \leqslant v-3 \qquad \text{and} \qquad \sum m_i = \lambda\binom{v}{2},$$

then there exists a $P_M[\lambda, v]$.

Proof. For v odd, we define C_x as we did in Case 1 in the proof of Theorem 1. If v and λ are both even, C_x is defined as it is in Case 11.

We also define for v odd

$$T=\lambda C_0+\lambda C_1+\lambda C_2+\cdots+\lambda C_{(v-1)/2},$$

and for v and λ even

$$T=\frac{\lambda}{2}C_0+\frac{\lambda}{2}C_1+\frac{\lambda}{2}C_2+\cdots+\frac{\lambda}{2}C_{v-2}.$$

Now T covers K_v λ times and $d(T)=v-2$. Thus, cutting T into segments $P_{m_1}, P_{m_2}, \dots, P_{m_k}$, we obtain a $P_M[\lambda, v]$.

The conjecture is still open, if some of the m_i-s equal $v-2$ or $v-1$, and for the case when λ is odd while v is even.

Acknowledgments

This paper is part of the author's Ph. D. Thesis which was written, under the guidance of Professor Haim Hanani, in Department of Mathematics, Technion, Israel Institute of Technology, Haifa, Israel. The author wishes to thank Professor Hanani for his vital help during the struggle with some difficult aspects of the problem.

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